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Classification of Ricci tensor in space–times with a four-parameter group of motions acting on null hypersurfaces

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Abstract. The Plebanski type of the Ricci tensor is given for all Bianchi type metrics admitting a four-parameter group of motions acting on null hypersurfaces using the Ludwig–Scanlan classification method and the Newman–Penrose formalism. A class of solutions is found for which the Ricci tensor has a null eigenvector.

1. Introduction

In the preceding paper (Abdel-Megied 1980) a classification method for the Ricci tensor (Ludwig and Scanlan 1971) together with the Newman–Penrose formalism (Newman and Penrose 1962) were used to determine the Plebanski types of the Ricci tensor in space–times with a four-parameter group of motions G_4 acting on non-null hypersurfaces.

In this paper we extend our study to the case where G_4 is acting on null hypersurfaces. As we know, the isotropy group belonging to G_4 having three-dimensional orbits as minimal invariant varieties must be a one-parameter group (Eisenhart 1961, Ehlers and Kundt 1962). Consequently, the metric with such G_4 can be only one of the Petrov types D, N or O (Ehlers and Kundt 1962).

We shall use here also the Newman–Penrose formalism to obtain the components of the trace-free Ricci tensor. After a comparison with the classification method of Ludwig and Scanlan (1971), here referred to as LS, and Hall (1976), one can determine the Plebanski type (Plebanski 1964) and decide whether the Ricci tensor has null or non-null eigenvectors (Hall 1976) (see also Crade and Hall (1979)). In § 2, we shall calculate the spin coefficients for each type of metric using the complex vectorial formalism (Cahen *et al* 1967, Israel 1970). In § 3, the Newman–Penrose equations, here referred to as NP equations, will be used to determine the Petrov type corresponding to each Bianchi-type metric and to get the trace-free components of the Ricci tensor; then, after a comparison with LS, the Plebanski type can be determined.

Throughout this paper, the signature of the space–time, in a local Lorentz frame, is -2, Greek indices with values 0, 1, 2, 3 are coordinate indices, while Latin letters are tetrad indices.

2. Spin coefficients and space–times having a four-parameter group of motions acting on null hypersurfaces

All metrics admitting a four-parameter group of motions acting on null hypersurfaces were given by Petrov (1969). Lauten and Ray (1977) investigated these metrics. As a result, we shall study all physical metrics with at least a four-parameter group of motions G_4.

These metrics are of the following Bianchi types:

$$(G_4\,\mathrm{I}_1) \qquad ds^2 = A^2(t)\,e^{-2x^3}(2dt\,dx^1 - dx^{22}) - B^2(t)\,dx^{32}, \tag{2.1}$$

$$(G_4\,\mathrm{III}) \qquad -ds^2 = 2dt\,dx^1 - 2x^3\,dx^2\,dx^3 + A^2(t)(dx^{22} + dx^{32}), \tag{2.2}$$

$$(G_4\,\mathrm{V}) \qquad ds^2 = 2dt\,dx^1 - A^2(t)\,e^{-x^1}(dx^{22} + dx^{32}), \tag{2.3}$$

$$(G_4\,\mathrm{VI}_2) \qquad ds^2 = 2dt\,dx^1 - A^2(t)(dx^{22} + dx^{32}), \tag{2.4}$$

$$(G_4\,\mathrm{VI}_3) \qquad -ds^2 = 2dt\,dx^1 + A^2(t)\,dx^{22} + 2B(t)\,dx^2\,dx^3 + C^2(t)\,dx^{32}, \tag{2.5}$$

$$(G_4\,\mathrm{VII}_1) \qquad ds^2 = 2dt\,dx^1 - A^2(t)(e^{-2x^3}\,dx^{22} + dx^{32}), \tag{2.6}$$

$$(G_4\,\mathrm{VII}_2) \qquad -ds^2 = 2dt\,dx^1 - 2e^{-x^3}\,dt\,dx^2 + A^2(t)(e^{-2x^3}\,dx^{22} + dx^{32}), \tag{2.7}$$

$$(G_4\,\mathrm{VIII}_1) \qquad ds^2 = 2dt\,dx^1 - A^2(t)(\cos^2 x^3\,dx^{22} + dx^{32}), \tag{2.8}$$

$$(G_4\,\mathrm{VIII}_2) \qquad -ds^2 = 2dt\,dx^1 - 2\sin x^3\,dt\,dx^2 + A^2(t)(\cos^2 x^3\,dx^{22} + dx^{32}). \tag{2.9}$$

For each type of the above metrics, we construct a complex null tetrad l^μ, n^μ, m^μ, $\bar{m}^\mu$ (Sachs 1961) as follows.

$$(G_4\,\mathrm{I}_1) \qquad l^\mu = \delta^\mu_1, \qquad n^\mu = e^{2x^3}\,\delta^\mu_0/A^2,$$
$$m^\mu, \bar{m}^\mu = (e^{x^3}\,\delta^\mu_2/A \pm i\,\delta^\mu_3/B)/\sqrt{2}, \tag{2.10}$$

$$(G_4\,\mathrm{III}) \qquad l^\mu = \delta^\mu_1, \qquad n^\mu = -\delta^\mu_0,$$
$$m^\mu, \bar{m}^\mu = [(1/A)(1 \pm ix^3/\Sigma)\delta^\mu_2 + (iA/\Sigma)\delta^\mu_3]/\sqrt{2}, \tag{2.11}$$

where $\Sigma^2 = A^4(t) - (x^3)^2 > 0$.

$$(G_4\,\mathrm{V}) \qquad l^\mu = \delta^\mu_0, \qquad n^\mu = \delta^\mu_1, \qquad m^\mu, \bar{m}^\mu = (e^{\frac{1}{2}x^1}/\sqrt{2}A)(\delta^\mu_2 \pm i\delta^\mu_3), \tag{2.12}$$

$$(G_4\,\mathrm{VI}_2) \qquad l^\mu = \delta^\mu_1, \qquad n^\mu = \delta^\mu_0, \qquad m^\mu, \bar{m}^\mu = (\delta^\mu_2 \pm i\delta^\mu_3)/\sqrt{2}A, \tag{2.13}$$

$$(G_4\,\mathrm{VI}_3) \qquad l^\mu = \delta^\mu_1, \qquad n = -\delta^\mu_0,$$
$$m^\mu, \bar{m}^\mu = [(1/A)(1 \pm iB/\Sigma)\delta^\mu_2 \mp (iA/\Sigma)\delta^\mu_3]/\sqrt{2}, \tag{2.14}$$

where $\Sigma^2 = A^2(t)C^2(t) - B^2(t) > 0$.

$$(G_4\,\mathrm{VII}_1) \qquad l^\mu = \delta^\mu_1, \qquad n^\mu = \delta^\mu_0, \qquad m^\mu, \bar{m}^\mu = e^{x^3}(\delta^\mu_2 \pm i\delta^\mu_3)/\sqrt{2}A, \tag{2.15}$$

$$(G_4\,\mathrm{VII}_2) \qquad l^\mu = \delta^\mu_1, \qquad n^\mu = -\delta^\mu_0, \qquad m^\mu, \bar{m}^\mu = (\delta^\mu_1 + e^{x^3}\,\delta^\mu_2 \pm i\delta^\mu_3)/\sqrt{2}A, \tag{2.16}$$

$$(G_4\,\mathrm{VIII}_1) \qquad l^\mu = \delta^\mu_1, \qquad n^\mu = d^\mu_0, \qquad m^\mu, \bar{m}^\mu = (\sec x^3 \delta^\mu_2 \pm i\delta^\mu_3)/\sqrt{2}A, \tag{2.17}$$

$$(G_4\,\mathrm{VIII}_2) \qquad l^\mu = \delta^\mu_1, \qquad n^\mu = -\delta^\mu_0,$$
$$m^\mu, \bar{m}^\mu = (-\tan x^3\delta^\mu_1 + \sec x^3\delta^\mu_2 \pm i\delta^\mu_3)/\sqrt{2}A, \tag{2.18}$$

where from now on $A = A(t)$, $B = B(t)$ and differentiation with respect to t will be denoted by a dot.

In the above construction l^μ is chosen, except for G_4 V, so as to coincide with one of the generators of the group (Petrov 1969). Thus we have, in each type except for G_4 V, that the vector l^μ is a null Killing vector.

The corresponding one-forms θ^a can be determined from the relations

$$dx^\mu = l^\mu\theta^0 + n^\mu\theta^1 - m^\mu\theta^2 - \bar{m}^\mu\theta^3. \tag{2.19}$$

In terms of θ^a the metric takes the form

$$ds^2 = 2(\theta^0\theta^1 - \theta^2\theta^3). \tag{2.20}$$

Now, the two-forms

$$Z^1 = \theta^3 \wedge \theta^0, \qquad Z^2 = \theta^1 \wedge \theta^2, \qquad Z^3 = \tfrac{1}{2}(\theta^1 \wedge \theta^0 - \theta^2 \wedge \theta^3) \tag{2.21}$$

form a basis for the space of self dual two-forms (Cahen *et al* 1967). The exterior differential of these basic two-forms is given by

$$\begin{aligned} dZ^1 &= -\tfrac{1}{2}\sigma_3 \wedge Z^1 - \sigma_2 \wedge Z^3, \\ dZ^2 &= \tfrac{1}{2}\sigma_3 \wedge Z^2 + \sigma_1 \wedge Z^3, \\ dZ^3 &= \tfrac{1}{2}\sigma_1 \wedge Z^1 - \tfrac{1}{2}\sigma_2 \wedge Z^2, \end{aligned} \tag{2.22}$$

where σ_1, σ_2 and σ_3 are three complex one-forms.

Writing these three forms in terms of the basic one-forms θ^a one obtains

$$\sigma_p = \sigma_{pa}\theta^a, \qquad p = 1, 2, 3. \tag{2.23}$$

The quantities σ_{pa} are called the Ricci rotation coefficients and are equivalent to the twelve Newman–Penrose spin coefficients (Newman and Penrose 1962). The relation between the NP spin coefficients and σ_{pa} is known (Israel 1970).

We give now the one-forms corresponding to each type of metric (2.1)–(2.18). Using (2.21), (2.22), (2.23) and the relations between the σ_{pa} and the NP spin coefficient (Israel 1970), we obtain, after somewhat lengthy calculations, the following results.

$(G_4\,\mathrm{I}_1)$

$$\theta^0 = dx^1, \qquad \theta^1 = A^2\,e^{-2x^3}\,dt,$$
$$\sqrt{2}\theta^2 = -(A\,e^{-x^3}\,dx^2 + i\,dx^3), \qquad \theta^3 = \bar{\theta}^2 \tag{2.24}$$

and

$$\kappa = \sigma = \rho = \epsilon = \alpha = \nu = \gamma = 0, \qquad \beta = -\pi = -\tau = i/\sqrt{2}B,$$
$$\lambda = (e^{2x^3}/2A^2)(B/B - \dot{A}/A), \qquad \mu = -(e^{2x^3}/2A^2)(A/A + \dot{B}/B). \tag{2.25}$$

$(G_4\,\mathrm{III})$

$$\theta^0 = dx^1, \qquad \theta^1 = -dt, \qquad \sqrt{2}\theta^2 = -A[dx^2 - (1/A^2)(x^3 - i\Sigma)\,dx^3],$$
$$\theta^3 = \bar{\theta}^2, \qquad \Sigma^2 = A^4 - (x^3)^2 > 0, \tag{2.26}$$

and

$$\begin{aligned} &\kappa = \sigma = \rho = \epsilon = \pi = \tau = \alpha = \beta = \nu = 0, \\ &\lambda = (A/A)(A^4/\Sigma^2 - ix^3/\Sigma), \qquad \mu = -(A/A)(x^3/\Sigma)^2, \\ &\gamma = \tfrac{1}{2}i(\dot{A}/A)(x^3/\Sigma). \end{aligned} \tag{2.27}$$

$(G_4\,\mathrm{V})$
$$\theta^0 = \mathrm{d}t, \qquad \theta^1 = \mathrm{d}x^1,$$
$$\sqrt{2}\theta^2 = -A\,\mathrm{e}^{-x^1}(\mathrm{d}x^2 + \mathrm{i}\mathrm{d}x^3), \qquad \theta^3 = \bar{\theta}^2, \tag{2.28}$$

and

$$\kappa = \sigma = \epsilon = \pi = \tau = \alpha = \beta = \lambda = \nu = \gamma = 0,$$
$$\rho = A/A, \qquad \mu = \tfrac{1}{2}. \tag{2.29}$$

$(G_4\,\mathrm{VI}_2)$
$$\theta^0 = \mathrm{d}x^1, \qquad \theta^1 = \mathrm{d}t, \qquad \sqrt{2}\theta^2 = -A(\mathrm{d}x^2 + \mathrm{i}\,\mathrm{d}x^3), \qquad \theta^3 = \bar{\theta}^2, \tag{2.30}$$

and

$$\kappa = \sigma = \rho = \epsilon = \pi = \tau = \alpha = \beta = \lambda = \nu = \gamma = 0,$$
$$\mu = -A/A. \tag{2.31}$$

$(G_4\,\mathrm{VI}_3)$
$$\theta^0 = \mathrm{d}x^1, \qquad \theta^1 = -\mathrm{d}t, \qquad \sqrt{2}\theta^2 = -A[\mathrm{d}x^2 - (1/A)(B + \mathrm{i}\Sigma)\,\mathrm{d}x^3],$$
$$\theta^3 = \bar{\theta}^2, \qquad \Sigma^2 = A^2C^2 - B^2, \tag{2.32}$$

and

$$\kappa = \sigma = \rho = \epsilon = \pi = \tau = \alpha = \beta = \mu = \nu = 0,$$
$$\lambda = -\tfrac{1}{2}[(\ln \Sigma/A^2)^{\cdot} + (\mathrm{i}B/\Sigma)(\ln A^2/B)^{\cdot}],$$
$$\gamma = -\tfrac{1}{4}\mathrm{i}(B/\Sigma)(\ln A^2/B)^{\cdot} \tag{2.33}$$

$(G_4\,\mathrm{VII}_1)$
$$\theta^0 = \mathrm{d}x^1, \qquad \theta^1 = \mathrm{d}t,$$
$$\sqrt{2}\theta^2 = -A(\mathrm{e}^{-x^3}\,\mathrm{d}x^2 + \mathrm{i}\,\mathrm{d}x^3), \qquad \theta^3 = \bar{\theta}^2 \tag{2.34}$$

and

$$\kappa = \sigma = \rho = \epsilon = \pi = \tau = \lambda = \gamma = \nu = 0,$$
$$\alpha = \beta = \mathrm{i}/2\sqrt{2}A, \qquad \mu = -A/A. \tag{2.35}$$

$(G_4\,\mathrm{VII}_2)$
$$\theta^0 = \mathrm{d}x^1 - \mathrm{e}^{-x^3}\,\mathrm{d}x^2, \qquad \theta^1 = -\mathrm{d}t,$$
$$\sqrt{2}\theta^2 = -A(\mathrm{e}^{-x^3}\,\mathrm{d}x^2 + \mathrm{i}\,\mathrm{d}x^3), \qquad \theta^3 = \bar{\theta}^2 \tag{2.36}$$

and

$$\kappa = \sigma = \rho = \epsilon = \pi = \tau = \lambda = \nu = 0,$$
$$\alpha = \beta = \mathrm{i}/2\sqrt{2}A, \qquad \mu = A/A + \mathrm{i}/2A^2, \qquad \gamma = \mathrm{i}/4A^2. \tag{2.37}$$

$(G_4\,\mathrm{VIII}_1)$
$$\theta^0 = \mathrm{d}x^1, \qquad \theta^1 = \mathrm{d}t, \qquad \sqrt{2}\theta^2 = -A\,(\cos x^3\,\mathrm{d}x^2 + \mathrm{i}\,\mathrm{d}x^3),$$
$$\theta^3 = \bar{\theta}^2 \tag{2.38}$$

and

$$\kappa = \sigma = \rho = \epsilon = \pi = \tau = \lambda = \gamma = \nu = 0,$$
$$\alpha = \beta = \mathrm{i}\tan x^3/2\sqrt{2}A, \qquad \mu = -A/A. \tag{2.39}$$

$(G_4\,\mathrm{VIII}_2)$
$$\theta^0 = \mathrm{d}x^1 + \sin x^3\,\mathrm{d}x^2, \qquad \theta^1 = -\mathrm{d}t,$$
$$\sqrt{2}\theta^0 = -A(\cos x^3\,\mathrm{d}x^2 + \mathrm{i}\,\mathrm{d}x^3), \qquad \theta^3 = \bar{\theta}^2 \tag{2.40}$$

and

$$\kappa = \sigma = \rho = \epsilon = \pi = \tau = \lambda = \nu = 0,$$
$$\alpha = \beta = \mathrm{i} \tan x^3/2\sqrt{2}A, \qquad \mu = A/A + \mathrm{i}/2A^2, \qquad \gamma = \mathrm{i}/4A^2. \tag{2.41}$$

From these results, we obtain the following geometrical properites for the null tetrad used here (Sachs 1961, Israel 1970).

(1) For all metrics considered above, the congruences l^μ are null geodesics parametrised by an affine parameter.

(2) Except for G_4 V, the congruence is non-diverging, twist- and shear-free with l^μ as a null Killing vector. In G_4 V, the congruence is twist- and shear-free but it is expanding.

(3) Except in G_4 I_1, the tetrad is parallelly propagated along l^μ.

3. Newman–Penrose equations and the Plebanski type of the Ricci tensor

Using the results of § 2, we can now insert the calculated spin coefficients for each Bianchi type of metric (2.1)–(2.9) in the NP equations (Newman and Penrose 1962, Pirani 1965). We write the intrinsic derivatives occurring in the NP equations as

$$D\varphi = \varphi_{;\mu} l^\mu, \qquad \Delta\varphi = \varphi_{;\mu} n^\mu, \qquad \delta\varphi = \varphi_{;\mu} m^\mu, \qquad \bar{\delta}\varphi = \varphi_{;\mu} \bar{m}^\mu. \tag{3.1}$$

From the NP equations, we obtain, for each Bianchi type, the following results.

(G_4 I_1)

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, \qquad \Psi_4 = \bar{\delta}\mu - \Delta\lambda + 3\mu\lambda,$$
$$\Lambda = R/24 = -1/2B^2,$$

where R is the scalar curvature.

$$\phi_{00} = \phi_{01} = \phi_{02} = \phi_{11} = 0,$$
$$\phi_{12} = \phi_{21} = -\mathrm{i}(\mathrm{e}^{2x^3}/\sqrt{2}BA^2)(B/B), \qquad \phi_{22} = -\Delta\mu - 2\mu^2 - \lambda^2.$$

So, we have the following results.

The metric G_4 I_1 is of Petrov type N. If B = constant, we have the Lauten–Ray solution for a null fluid (Lauten and Ray 1977). In this case, we find that ϕ_{22} is the only non-zero component of the trace-free Ricci tensor and after a comparison with LS (Ludwig and Scanlan 1971), we conclude that the Ricci tensor has the Plebanski type $[4N]_{[2]}$ with Segre characteristic [(211)].

(G_4 III)

$$\Psi_0 = \Psi_1 = \Psi_2 = 0. \qquad \Lambda = 0$$
$$\delta\lambda - \delta\mu = -\Psi_3 + \phi_{21}, \qquad \Delta\lambda = -2\mu\lambda - (3\gamma - \bar{\gamma})\lambda - \Psi_4,$$
$$\phi_{00} = \phi_{01} = \phi_{02} = \phi_{11} = 0, \qquad -\Delta\mu = \mu^2 + \lambda\bar{\lambda} + \phi_{22},$$
$$\phi_{12} = -(\dot{A}/2\sqrt{2})(A/\Sigma)^4, \qquad -\Psi_3 = -(\dot{A}/2)\sqrt{2}(A/\Sigma)^4,$$
$$\Sigma = [A^4 - (x^3)^2]^{1/2}.$$

Now, since ϕ_{12} must be complex, it follows from the last two equations that A must be constant, which implies that $\Psi_3 = 0$. With A = constant, we see from (2.27) that all the spin coefficients vanish. Consequently, we get $\Psi_4 = 0$. From these results, one can easily see that the metric G_4 III is flat. The transformation

$$\sqrt{2}\tilde{t} = x^1 - t, \qquad \sqrt{2}\tilde{x}^1 = x^1 + t, \qquad \tilde{x}^2 = Kx^2 - (x^3/K)^2,$$
$$2\tilde{x}^3 = x^3[K^2 - (x^3/K)^2]^{1/2} + K\sin^{-1}(x^3/K^2), \qquad K = A = \text{constant}$$

takes the metric G_4 III into the Minkowski metric

$$ds^2 = d\tilde{t}^2 - d\tilde{x}^{1^2} - d\tilde{x}^{2^2} - d\tilde{x}^{3^2}.$$

$(G_4\,\mathrm{V})$ $\quad \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0, \qquad \Lambda = -A/4A,$

and

$$\phi_{00} = (A/A)^{\cdot} - (\dot{A}/A)^2, \qquad \phi_{11} = \Lambda, \qquad \phi_{22} = -\tfrac{1}{4}, \qquad \phi_{01} = \phi_{02} = \phi_{12} = 0.$$

Thus the metric G_4 V is conformally flat. To obtain the conformal factor, we introduce new coordinates $\tilde{t} = \int (1/A)\,dt$, $\tilde{x}^1 = e^{x^1}$, $\tilde{x}^2 = x^2$, $\tilde{x}^3 = x^3$, so the metric takes the form

$$ds^2 = (\tilde{A}^2(\tilde{t})/\tilde{x}^1)(2d\tilde{t}\,d\tilde{x}^1 - d\tilde{x}^{2^2} - d\tilde{x}^{3^2}).$$

From the above results, we see that the non-zero components of the trace-free Ricci tensor are ϕ_{00}, ϕ_{11}, ϕ_{22} and after a comparison with LS, we conclude that the Ricci tensor has no null eigenvector. A calculation for the eigenvalues shows that the Ricci tensor has the Plebanski type $[T - S_1 - 2S_2]_{[1-1-1]}$, with Segre characteristic $[(11)11]$.

$(G_4\,\mathrm{VI}_2)$ $\quad \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0, \qquad \Lambda = 0,$

and

$$\phi_{00} = \phi_{01} = \phi_{02} = \phi_{11} = \phi_{12} = 0, \qquad \phi_{22} = (\dot{A}/A)^{\cdot} - (\dot{A}/A)^2.$$

Thus the metric G_4 VI_2 is conformally flat. With the new time-coordinate $\tilde{t} = \int A(t)\,dt$, the metric takes the form

$$ds^2 = \tilde{A}^2(\tilde{t})(2d\tilde{t}\,dx^1 - dx^{2^2} - dx^{3^2}).$$

Now, since ϕ_{22} is the only non-zero component of the trace-free Ricci tensor, then it follows after a comparison with LS that the Ricci tensor has the Plebanski type $[4N]_{[2]}$ with Segre characteristic $[(211)]$.

$(G_4\,\mathrm{VI}_3)$ $\quad \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, \qquad \Lambda = 0,$

$$\Psi_4 = \lambda + (i\lambda B/2\Sigma)(\ln A^2/B)^{\cdot}, \qquad \Sigma = (A^2C^2 - B^2)^{1/2}$$

and

$$\phi_{00} = \phi_{01} = \phi_{02} = \phi_{11} = \phi_{12} = 0, \qquad \phi_{22} = \lambda\dot{\lambda} \neq 0.$$

From these equations we conclude that the metric G_4 VI_3 is of Petrov type N and the Ricci tensor has the Plebanski type $[4N]_{[2]}$ with Segre characteristic $[(211)]$.

$(G_4\,\mathrm{VII}_1)$ $\quad \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \qquad \Psi_2 = -2\Lambda = 1/6A^2,$

and

$$\phi_{00} = \phi_{01} = \phi_{02} = \phi_{12} = 0, \qquad \phi_{11} = 3\Lambda,$$
$$\phi_{22} = -\Delta\mu - \mu^2, \qquad \Delta\alpha = -\mu\alpha.$$

From (2.34) and the last two equations we see that A must be constant. Consequently, μ must vanish, which implies that $\phi_{22} = 0$. So, we conclude that the metric G_4 VII_1 is of type D. Now, since ϕ_{11} is the only non-zero component of the trace-free Ricci tensor, then it follows from a comparison with LS that the Ricci tensor has the Plebanski type $[2T\text{-}2S]_{[1-1]}$ with Segre characterisitc [(11)(11)].

(G_4 VII$_2$) $\Psi_0=\Psi_1=\Psi_3=\Psi_4=0,\qquad \Psi_2=-2\Lambda=1/6A^2,$

and

$$\phi_{00}=\phi_{01}=\phi_{02}=\phi_{12}=0,\qquad \phi_{11}=3\Lambda,$$
$$\phi_{22}=-\Delta\mu-\mu^2=(A/A)^{\cdot}-(\dot{A}/A)^2+1/4A^4-2\mathrm{i}\dot{A}/A^3.$$

Since ϕ_{22} must be real, then A must be constant. Now, the only non-zero components of the trace-free Ricci tensor are ϕ_{11} and ϕ_{22}. A comparison with LS shows that the Ricci tensor has the Plebanski type $[2N-2S]_{[2-1]}$ with Segre characteristic [2(11)]. The metric of the Bianchi type is also of Petrov type D.

(G_4 VIII$_1$) $\Psi_0=\Psi_1=\Psi_3=\Psi_4=0,\qquad \Psi_2=-2\Lambda=(\tan^2x^3)/6A^2,$

and

$$\phi_{00}=\phi_{01}=\phi_{02}=\phi_{12}=0,\qquad \phi_{11}=3\Lambda,$$
$$\phi_{22}=-\Delta\mu-\mu^2,\qquad \Delta\alpha=-\mu\alpha.$$

From (2.38) and the last two equations we see that A must be constant. Consequently, $\mu=0$, which implies that $\phi_{22}=0$. So, we have the same results obtained in G_4 VII$_1$.

(G_4 VIII$_2$) $\Psi_0=\Psi_1=\Psi_3=\Psi_4=0,\qquad \Psi_2=-2\Lambda=(1/4A^2)(\sec^2x^3+\tan^2x^3),$

and

$$\phi_{00}=\phi_{01}=\phi_{02}=\phi_{12}=0,\qquad \phi_{11}=4\Lambda,$$
$$\phi_{22}=-\Delta\mu-\mu^2=(A/A)^{\cdot}-(\dot{A}/A)^2+\tfrac{1}{4}A^4-2\mathrm{i}\dot{A}/A.$$

As we have seen in G_4 VII, A must be constant, since ϕ_{22} must be real. So, we have the same results obtained in G_4 VII$_2$.

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